

Spectral geometry and quantum gravity

Giampiero Esposito

INFN, Sezione di Napoli, Mostra d'Oltremare Padiglione 20, 80125 Napoli, Italy

Abstract

Recent progress in quantum field theory and quantum gravity relies on mixed boundary conditions involving both normal and tangential derivatives of the quantized field. In particular, the occurrence of tangential derivatives in the boundary operator makes it possible to build a large number of new local invariants. The integration of linear combinations of such invariants of the orthogonal group yields the boundary contribution to the asymptotic expansion of the integrated heat-kernel. This can be used, in turn, to study the one-loop semiclassical approximation. The coefficients of linear combination are now being computed for the first time. They are universal functions, in that are functions of position on the boundary not affected by conformal rescalings of the background metric, invariant in form and independent of the dimension of the background Riemannian manifold. In Euclidean quantum gravity, the problem arises of studying infinitely many universal functions.

In classical and quantum field theory, as well as in the current attempts to develop a quantum theory of the universe and of gravitational interactions, it remains very useful to describe physical phenomena in terms of differential equations for the variables of the theory, supplemented by boundary conditions for the solutions of such equations. For example, the problems of electrostatics, the analysis of waveguides, the theory of vibrating membranes, the Casimir effect, van der Waals forces, and the problem of how the universe could evolve from an initial state, all need a careful assignment of boundary conditions. In the latter case, if one follows a path-integral approach, one faces two formidable tasks: (i) the specification of the geometries occurring in the “sum over histories” and matching the assigned boundary data; (ii) the choice of boundary conditions on metric perturbations which may lead to the evaluation of the one-loop semiclassical approximation.

Indeed, while the full path integral for quantum gravity is a fascinating idea but remains a formal tool, the one-loop calculation may be put on solid ground, and appears particularly interesting because it yields the first quantum corrections to the underlying classical theory (although it is well known that quantum gravity based on Einstein’s theory is not perturbatively renormalizable). Within this framework, it is of crucial importance to evaluate the one-loop divergences of the theory under consideration. Moreover, the task of the theoretical physicist is to understand the deeper general structure of such divergences. For this purpose, one has to pay attention to all geometric invariants of the problem, in a way made clear by a branch of mathematics known as invariance theory [1]. The key geometric elements of our problem are hence as follows.

A Riemannian geometry (M, g) is given, where the manifold M is compact and has a boundary ∂M with induced metric γ , and the metrics g and γ are positive-definite. A vector bundle over M , say V , is given, and one has also to consider a vector bundle \tilde{V} over ∂M . An operator of Laplace type, say P , is a second-order elliptic operator with leading symbol given by the

metric. Thus, one deals with a map from the space of smooth sections of V onto itself,

$$P : C^\infty(V, M) \rightarrow C^\infty(V, M), \quad (1)$$

which can be expressed in the form

$$P = -g^{ab} \nabla_a^V \nabla_b^V - E, \quad (2)$$

where ∇^V is the connection on V , and E is an endomorphism of V : $E \in \text{End}(V)$. Moreover, the boundary operator is a map

$$\mathcal{B} : C^\infty(V, M) \rightarrow C^\infty(\tilde{V}, \partial M), \quad (3)$$

and contains all the informations on the boundary conditions of the problem. Since we are interested in a generalization of Robin boundary conditions [2–9], we consider a boundary operator of the form (the operation of restriction to the boundary being implicitly understood)

$$\mathcal{B} = \nabla_N + \frac{1}{2} [\Gamma^i \hat{\nabla}_i + \hat{\nabla}_i \Gamma^i] + S. \quad (4)$$

With our notation, ∇_N is the normal derivative operator $\nabla_N \equiv N^a \nabla_a$ (N^a being the inward-pointing normal to ∂M), S is an endomorphism of the vector bundle \tilde{V} , Γ^i are endomorphism-valued vector fields on ∂M , and $\hat{\nabla}_i$ denotes tangential covariant differentiation with respect to the connection induced on ∂M . More precisely, when sections of bundles built from V are involved, $\hat{\nabla}_i$ means

$$\nabla_{\partial M}^{(\text{lc})} \otimes 1 + 1 \otimes \nabla,$$

where $\nabla_{\partial M}^{(\text{lc})}$ denotes the Levi-Civita connection of the boundary of M . Hereafter, we assume that $1 + \Gamma^2 > 0$, to ensure strong ellipticity of the boundary-value problem [9].

The case of mixed boundary conditions corresponds to the possibility of splitting the bundle V , in a neighbourhood of ∂M , as the direct sum of two bundles, say V_1 and V_2 , for each of which a boundary operator of the Dirichlet or (generalized) Robin type is also given. The former involves a projection operator, say Π , while the latter may also involve the complementary projector, $1 - \Pi$, and the metric of V , say H [5]:

$$\mathcal{B}_1 = \Pi, \quad (5)$$

$$\mathcal{B}_2 = (1 - \Pi) \left[H \nabla_N + \frac{1}{2} (\Gamma^i \hat{\nabla}_i + \hat{\nabla}_i \Gamma^i) + S \right]. \quad (6)$$

We can now come back to our original problem, where only the boundary operator (4) occurs, and investigate its effect on heat-kernel asymptotics [8,9]. Indeed, given the heat equation for the operator P , its kernel, i.e. the heat kernel, is, by definition, a solution for $t > 0$ of the equation

$$\left(\frac{\partial}{\partial t} + P \right) U(x, x'; t) = 0, \quad (7)$$

jointly with the initial condition

$$\lim_{t \rightarrow 0} \int_M dx' \sqrt{\det g(x')} U(x, x'; t) \rho(x') = \rho(x), \quad (8)$$

and the boundary condition

$$[\mathcal{B}U(x, x'; t)]_{\partial M} = 0. \quad (9)$$

The fibre trace of the heat-kernel diagonal, i.e. $\text{Tr}U(x, x; t)$, admits an asymptotic expansion which describes the *local* asymptotics, and involves interior invariants and boundary invariants. Interior invariants are built universally and polynomially from the metric, the Riemann curvature R^a_{bcd} of M , the bundle curvature, say Ω_{ab} , the endomorphism E , and their covariant derivatives. By virtue of Weyl's work on the invariants of the orthogonal group, these polynomials can be found by using only tensor products and contraction of tensor arguments [7,10]. The asymptotic expansion of the integral

$$\int_M dx \sqrt{\det g} \text{Tr}U(x, x; t) \equiv \text{Tr}_{L^2}(e^{-tP}), \quad (10)$$

yields instead the *global* asymptotics. For our purposes, it is more convenient to weight e^{-tP} with a smooth function $f \in C^\infty(M)$, and then consider the asymptotic expansion

$$\text{Tr}_{L^2}(f e^{-tP}) \equiv \int_M dx \sqrt{\det g} f(x) \text{Tr}U(x, x; t) \sim (4\pi t)^{-m/2} \sum_{l=0}^{\infty} t^{l/2} A_{l/2}(f, P, \mathcal{B}). \quad (11)$$

Following Ref. [8], m is the dimension of M , and the coefficient $A_{l/2}(f, P, \mathcal{B})$ consists of an interior part, say $C_{l/2}(f, P)$, and a boundary part, say $B_{l/2}(f, P, \mathcal{B})$. The interior part vanishes for all odd values of l , whereas the boundary part only vanishes if $l = 0$. The interior part is obtained by integrating over M the linear combination of local invariants of the appropriate dimension mentioned above, where the coefficients of the linear combination are *universal constants*, independent of m . Moreover, the boundary part is obtained upon integration over ∂M of another linear combination of local invariants. In that case, however, the structure group is $O(m-1)$ [10], and the coefficients of linear combination are *universal functions* [8], independent of m , unaffected by conformal rescalings of g , and invariant in form (i.e. they are functions of position on the boundary, whose form is independent of the boundary being curved or totally geodesic). It is thus clear that the general form of the $A_{l/2}$ coefficient is a well posed problem in invariance theory, where one has to take all possible local invariants built from $f, R^a_{bcd}, \Omega_{ab}, K_{ij}, E, S, \Gamma^i$ and their covariant derivatives (hereafter, K_{ij} is the extrinsic-curvature tensor of the boundary), eventually integrating their linear combinations over M and ∂M . For example, in the boundary part $B_{l/2}(f, P, \mathcal{B})$, the local invariants integrated over ∂M are of dimension $l-1$ in tensors of the same dimension of the second fundamental form of the boundary, for all $l \geq 1$ [1,10]. The universal functions associated to all such invariants can be found by using the conformal-variation method described in Refs. [1,7,8,10], jointly with the analysis of simple examples and particular cases.

In other words, recurrence relations of algebraic nature exist among all universal functions, and one can therefore use the solutions of simple problems to determine completely the remaining set of universal functions for a given value of the integer l in the asymptotic expansion (11). The detailed investigation of the coefficients $A_1, A_{3/2}$ and A_2 when the boundary operator is given by Eq. (4) and all curvature terms are non-vanishing is performed in Ref. [8]. One then finds the result (which holds for all integer values of $l \geq 2$)

$$A_{l/2}(f, P, \mathcal{B}) = \tilde{A}_{l/2}(f, P, \mathcal{B}) + \int_{\partial M} \text{Tr} \left[a_{l/2}(f, R, \Omega, K, E, \Gamma, S) \right], \quad (12)$$

where $\tilde{A}_{l/2}(f, P, \mathcal{B})$ is formally analogous to the purely Robin case, but replacing the universal constants in the boundary terms with universal functions, whereas $a_{l/2}$ is a linear combination of all local invariants of the given dimension which involve contractions with Γ^i . Our task is now

to derive an algorithm for the general form of $a_{l/2}$, since it helps a lot to have a formula that clarifies the general features of a scheme where the number of new invariants is rapidly growing. Indeed, from Ref. [8], we know that, in a_1 , only one new invariant occurs: $f K_{ij} \Gamma^i \Gamma^j$, whereas in $a_{3/2}$ 11 new invariants occur, obtained by contraction of Γ^i with terms like (tensor indices are here omitted for simplicity)

$$f K^2, f K S, f \widehat{\nabla} K, f \widehat{\nabla} S, f R, f \Omega, f_{;N} K.$$

In a_2 , the number of new invariants is 68: 57 involve contractions of Γ^i with terms like

$$f K^3, f K^2 S, f K S^2, f R K, f \Omega K, f E K, f R S, f \Omega S, f K \widehat{\nabla} K, f S \widehat{\nabla} K, \\ f K \widehat{\nabla} S, f S \widehat{\nabla} S, f \widehat{\nabla} \widehat{\nabla} K, f \widehat{\nabla} \widehat{\nabla} S, f \nabla R, f \nabla \Omega, f \nabla E,$$

10 local invariants involve contractions of Γ^i with contributions like

$$f_{;N} K^2, f_{;N} K S, f_{;N} \widehat{\nabla} K, f_{;N} \widehat{\nabla} S, f_{;N} R, f_{;N} \Omega,$$

and the last invariant is $f_{;NN} K_{ij} \Gamma^i \Gamma^j$ [8]. It is thus clear that the knowledge of all local invariants in $a_{l/2}$ plays a role in the form of $a_{(l+1)/2}$, and one can write the formulae

$$a_1 = f \sum_{i=1}^{i_1} \mathcal{U}_i^{(1,1)} I_i^{(1)}, \quad (13)$$

$$a_{3/2} = f \sum_{i=1}^{i_2} \mathcal{U}_i^{(3/2,3/2)} I_i^{(3/2)} + f_{;N} \sum_{i=1}^{i_1} \mathcal{U}_i^{(3/2,1)} I_i^{(1)}, \quad (14)$$

$$a_2 = f \sum_{i=1}^{i_3} \mathcal{U}_i^{(2,2)} I_i^{(2)} + f_{;N} \sum_{i=1}^{i_2} \mathcal{U}_i^{(2,3/2)} I_i^{(3/2)} + f_{;NN} \sum_{i=1}^{i_1} \mathcal{U}_i^{(2,1)} I_i^{(1)}. \quad (15)$$

With our notation, $i_1 = 1, i_2 = 10, i_3 = 57$, and $\mathcal{U}_i^{(x,y)}$ are the universal functions, where i is an integer ≥ 1 , x is always equal to the order $l/2$ of $a_{l/2}$, and y is equal to the label of the invariant $I_i^{(y)}$, which does not contain f or derivatives of f and is of dimension $2y - 1$ in K or in tensors of the same dimension of K .

These remarks make it possible to write down a formula which holds for all $l \geq 2$:

$$a_{l/2}(f, R, \Omega, K, E, \Gamma, S) = \sum_{r=0}^{l-2} f^{(r)} \sum_{i=1}^{i_{l-r-1}} \mathcal{U}_i^{(l/2, (l-r)/2)} [\Gamma^2] I_i^{(l-r)/2} [R, \Omega, K, E, \Gamma, S], \quad (16)$$

where $f^{(r)}$ is the normal derivative of f of order r (with $f^{(0)} = f$), and square brackets are used for the arguments of universal functions and local invariants, respectively. The equations (12) and (16) represent the desired parametrization of heat-kernel coefficients with generalized boundary conditions, provided that the Γ^i are covariantly constant [8].

One has now to evaluate the universal functions in the general formulae for $A_{3/2}, A_2, A_{5/2}$ and so on. For the coefficients $A_{3/2}$ and A_2 , results of a limited nature are available in Ref. [8], which show that all universal functions are generated from $\sqrt{1 + \Gamma^2}$ and $\frac{1}{\sqrt{-\Gamma^2}} \text{Arctanh} \sqrt{-\Gamma^2}$. Upon completion of this hard piece of work, one could perform the evaluation of all universal functions for $A_{5/2}(f, P, \mathcal{B})$ as well, possibly with the help of computers. For this purpose, one has to combine the conformal-variation method with the analysis of simpler cases. As shown in Refs.

[8,9], one then obtains a quicker and more elegant derivation of the coefficient $A_1(f, P, \mathcal{B})$. There are thus reasons to expect that, in the near future, all heat-kernel coefficients with generalized boundary conditions may be obtained via a computer algorithm in a relatively short time. This adds evidence in favour of the understanding of general mathematical structures being very helpful in providing the complete solution of difficult problems in physics and mathematics. In particular, from the point of view of quantum field theory in curved manifolds, this would mean an entirely geometric understanding of the first set of quantum corrections to the underlying classical theory, with the help of invariance theory [1], functorial methods [10] and computer programs.

In Euclidean quantum gravity, however, if one uses the de Donder gauge-averaging functional, and if one requires invariance of the whole set of boundary conditions under infinitesimal diffeomorphisms on metric perturbations, one finds boundary operators of the kind (5) and (6), where the matrix Γ^2 commutes with S *but not* with Γ^i [8]. This implies in turn that there exist infinitely many different tensors of the type [11]

$$T_{(m)}^{ij}(\Gamma^l) \equiv \text{Tr} \left[\alpha_{(m)}(\Gamma^2) \Gamma^i \beta_{(m)}(\Gamma^2) \Gamma^j \right],$$

which can contribute already to the integrand for $A_1(f, P, \mathcal{B})$, upon contraction with K_{ij} . Thus, for the boundary operator given by the direct sum of Eqs. (5) and (6), even the A_1 coefficient is unknown.

One thus faces a highly non-trivial problem. On the one hand, analytic results exist for the A_2 coefficient with boundary operator (5) and (6) in the particular case of a portion of flat Euclidean background bounded by a three-sphere [6,12]. Moreover, it has been shown in Ref. [5] that the boundary operator given by the direct sum of (5) and (6) leads to a symmetric operator on metric perturbations. However, in the non-commuting case relevant for Euclidean quantum gravity, even the building blocks of geometric invariants involving Γ^i are unknown. This is why it remains unclear how to write a general and unambiguous formula for heat-kernel coefficients. The solution of such a problem is of crucial importance in quantum gravity for the following reasons:

- (i) to improve the understanding of BRST invariant boundary conditions [13];
- (ii) to obtain an entirely geometric description of the one-loop divergences in quantum gravity and quantum supergravity [6];
- (iii) as a first step towards the quantization in arbitrary gauges on manifolds with boundary;
- (iv) to clarify the differences between Yang-Mills fields and the gravitational field;
- (v) to complete the application of the effective-action programme to perturbative quantum gravity.

Acknowledgments

The author is much indebted to Ivan Avramidi and Alexander Kamenshchik for scientific collaboration on the topics described in this paper.

References

- [1] Gilkey P. B. *Invariance Theory, The Heat Equation, and The Atiyah-Singer Index Theorem* (Chemical Rubber Company, Boca Raton, 1995).
- [2] Abouelsaood A. , Callan C. G. , Nappi C. R. and Yost S. A. Nucl. Phys. B **280**, 599 (1987).
- [3] Barvinsky A. O. Phys. Lett. B **195**, 344 (1987).
- [4] McAvity D. M. and Osborn H. Class. Quantum Grav. **8**, 1445 (1991).
- [5] Avramidi I. G. , Esposito G. and Kamenshchik A. Yu. Class. Quantum Grav. **13**, 2361 (1996).
- [6] Esposito G. , Kamenshchik A. Yu. and Pollifrone G. *Euclidean Quantum Gravity on Manifolds with Boundary (Fundamental Theories of Physics 85)* (Kluwer, Dordrecht, 1997).
- [7] Esposito G. *Dirac Operator and Spectral Geometry* (hep-th/9704016).
- [8] Avramidi I. G. and Esposito G. *New Invariants in the One-Loop Divergences on Manifolds with Boundary* (old version in hep-th/9701018).
- [9] Dowker J. S. and Kirsten K. *Heat-Kernel Coefficients for Oblique Boundary Conditions* (hep-th/9706129).
- [10] Branson T. P. and Gilkey P. B. Commun. Part. Diff. Eqs. **15**, 245 (1990).
- [11] Avramidi I. G. and Esposito G. *Universal Functions in Euclidean Quantum Gravity* (hep-th/9702150).
- [12] Esposito G. , Kamenshchik A. Yu. , Mishakov I. V. and Pollifrone G. Phys. Rev. D **52**, 3457 (1995).
- [13] Moss I. G. and Silva P. J. Phys. Rev. D **55**, 1072 (1997).